Midterm Exam – Take-Home Portion  
Complex Variables, Spring 2016

Solve each of the following problems on your own. You are not permitted to talk to ANYONE except Dr. Gill about this exam. Your solutions are due before 9:30 on Friday, May 6. You are permitted to use your book and class notes as much as you need to, but no other complex variables books or internet resources. Please be sure to turn in all of your work and to clearly justify your answers.

1. Where is \( f(z) = x^2 - y^2 + i2x|y| \) differentiable? Where is it analytic? Show all your work, clearly justifying and explaining your answers.

2. In this question, we will consider the function \( f(z) = (x + iy) = e^r \sin x + ie^r \cos x \).
   a. Show that the image of a horizontal line \( y = b \) under the mapping \( f(z) = e^r \sin x + ie^r \cos x \) is circle and find an equation for the circle. Is the line mapped one-to-one onto the circle (that is, does the image of the line cover the entire circle exactly once)? If not, what happens? If an object moves from left to right along the line \( y = b \), in what direction does its image travel around the circle? Clearly justify your answers.
   b. What is the image of a vertical line \( x = a \) under the mapping? Is this line mapped one-to-one onto its image? If not, what happens? If an object moves upward along the line \( x = a \), in what direction does its image travel? Clearly justify your answers.
   c. For what values of \( z \) is \( f \) differentiable? For those values of \( z \), what is the derivative of \( f \)?
   d. Verify that \( f(z) = ie^{-iz} \).

3. The polar form of the Cauchy-Riemann Equations
   a. A complex function \( f(z) = u(x,y) + iv(x,y) \) can be converted into polar form using the coordinate transformation \( x = r \cos \theta \) and \( y = r \sin \theta \). Use this transformation along with the chain rules
      \[
      \frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial r}
      \]
      (and similar chain rules for \( v \)) to show that
      \[
      u_r = u_x \cos \theta + u_y \sin \theta \quad \text{and} \quad u_\theta = -u_x r \sin \theta + u_y r \cos \theta
      \]
      \[
      v_r = v_x \cos \theta + v_y \sin \theta \quad \text{and} \quad v_\theta = -v_x r \sin \theta + v_y r \cos \theta.
      \]
   b. Use the results of (a) to prove that the standard Cauchy-Riemann equations (in terms of \( x \) and \( y \)) are equivalent to the following polar form of the Cauchy-Riemann equations:
      \[
      ru_r = v_\theta \quad \text{and} \quad rv_r = -u_\theta.
      \]
      (Note: to show equivalence you must prove that both (i) if the standard CR hold, then the polar CR also hold; and (ii) if the polar CR hold, then the standard CR must also hold.)
   c. We have previously proven that if \( f \) is analytic at a point \( z_0 = x_0 + iy_0 \), then
      \[
      f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0).
      \]
      Show that with \( z_0 = r_0 e^{i\theta_0} \), this can be rewritten as
      \[
      f'(z_0) = e^{-i\theta_0} \left[ u_r(r_0, \theta_0) + iv_r(r_0, \theta_0) \right].
      \]
Parts (b) and (c) above have provided a partial proof of the following theorem:

Theorem: Suppose that \( f(z) = u(r, \theta) + iv(r, \theta) \) is continuous in some neighborhood of the point \( z_0 = r_0 e^{i\theta_0} \) and that all of its partial derivatives \( u_r, u_\theta, v_r, \) and \( v_\theta \) are continuous at the point \( (r_0, \theta_0) \). Also assume that the polar Cauchy-Riemann equations \( ru_r = v_\theta \) and \( rv_r = -u_\theta \)
hold at \((r_0, \theta_0)\). Then \(f\) is differentiable at \(z_0\), and \(f'(z_0)\) can be computed by
\[
f'(z_0) = e^{-i\theta_0} \left[u_r(r_0, \theta_0) + iv_r(r_0, \theta_0)\right].
\]

4. Let \(F(z) = F\left(re^{i\theta}\right) = \ln r + i\theta\) where \(r > 0\) and \(-\pi < \theta \leq \pi\).

a. Briefly explain why \(F(x) = \ln x\) if \(x\) is any positive real number.

b. Note that \(F\) is defined for all \(z \neq 0\). Is it possible to define \(F(0)\) in such a way that \(F\) is continuous at \(z = 0\)? Why?

c. Show that \(F\) is an inverse of the exponential function. That is, show that
(i) \(e^{F(z)} = z\) for all \(z \neq 0\) and (ii) \(F(e^z) = z\) if \(-\pi < \text{Im } z < \pi\).

d. Use theorem stated at the end of question 3 above to show that \(F\) is analytic when \(r > 0\) and \(-\pi < \theta < \pi\). (That is, \(F\) is analytic everywhere except the origin and the negative real axis.)

Also show that \(F'(z) = 1/z\).

BONUS: Clearly explain why \(F\) is not analytic when \(\theta = \pi\). (the negative real axis)

Note: Parts (a), (c), and (d) indicate that \(F\) is a natural way to extend the definition of the natural logarithm \(\ln x\) to complex numbers.