Exercises 3.6, 3.30, and 3.31.

1. Prove the following trig identities for all complex numbers $z, z_1, z_2$. (Note that this is a proof of parts of Proposition 3.17 from the textbook, and is also part of Exercise 3.35. All of the standard trig identities in Proposition 3.17 carry over from the real trig functions and still hold for complex trig functions.)
   a. $\sin(-z) = -\sin z$
   b. $\cos(-z) = \cos z$
   c. $\sin(z + 2\pi) = \sin z$
   d. $\sin(z + \frac{\pi}{2}) = \cos z$
   e. $\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$

2. We showed in class that the image of a vertical line $x = a$ (with $a \neq 0$) under the mapping $w = 1/z$ is a circle with center $\frac{1}{2a}$ and radius $\frac{1}{2|a|}$.
   a. Determine the image of a horizontal line $y = b$ (with $b \neq 0$) under the mapping $w = 1/z$.
      What happens if $b = 0$? As the point $z$ moves horizontally from $-\infty$ to $\infty$ along the line $y = b$, describe how the image $w = 1/z$ moves through the $w$ plane.
   b. How does the answer to part (a) change if $b = 0$?
   c. Show that the image of the circle $\left\{ z : |z| + \frac{2i}{3} = \frac{4}{3} \right\}$ under the mapping $w = 1/z$ is the circle with radius 1 and center at $-i/2$.
   d. What is the image of $\left\{ z : |z + \frac{2i}{3}| < \frac{4}{3} \right\}$ under the mapping $w = 1/z$?

3. Let $M(z) = \frac{az + b}{cz + d}$ where $a, b, c, d$ are complex numbers with $c \neq 0$ and $ad - bc \neq 0$.
   Note that Proposition 3.3 from the textbook (which you proved in Exercise 6 from Chapter 3) implies that $M(z)$ can be decomposed into the following sequence of transformations:
   i. $w = z + \frac{d}{c}$
   ii. $\zeta = \frac{1}{w}$
   iii. $\psi = \frac{-ad - bc}{c^2} \zeta$
   iv. $M = \psi + \frac{a}{c}$
   a. For the mapping $f(z) = \frac{i - iz}{1 + z}$, what are the values of $a, b, c, d$? Write out the form of the 4 parts of the decomposition above for this function. Describe the
geometric effect of each step of the decomposition. Use the decomposition to explain
why this function maps the interior of the unit circle $D = \{ z : |z| < 1 \}$ onto the upper
half plane $\mathcal{H} = \{ w : \text{Im}(w) > 0 \}$.

b. Based on the decomposition above, explain in as much detail as possible why a
function of the form $M(z) = \frac{az + b}{cz + d}$ will always map generalized circles onto
generalized circles (it will probably be useful to consider some cases…). I’m not
looking for a formal proof here – just describe what each step in the decomposition
would do to a circle or a line….