Complex Variables: Homework #1

Read http://mathfaculty.fullerton.edu/mathews/c2003/ComplexNumberOrigin.html.

1. Compute the cube of \(2 - \sqrt{-1}\) to verify that it is a cube root of \(2 - 11\sqrt{-1}\).

2. Explain why cubic equations, rather than quadratic equations, played the pivotal role in helping to obtain the acceptance of complex numbers.

3. Find all solutions to \(27x^3 - 9x - 2 = 0\) by performing the following steps:
   i. Determine values of \(p\) and \(q\) such that this equation is equivalent to an equation of the form \(x^3 = 3px + 2q\).
   ii. Use the formula from slide #9 from the first day of class to find one solution to the equation.
   iii. Use the root found in (ii) to factor the original cubic into a linear term and a quadratic term. Find the roots (if any) of the quadratic term.

4. Use Bombelli’s techniques to find all solutions to \(x^3 - 87x - 130 = 0\). (For an example of similar computations, look at the example of Bombelli’s “wild thought” about midway through the webpage listed at the top of this assignment.)

5. A depressed cubic equation is a cubic equation which (i) has no \(x^2\) term and (ii) has a coefficient of 1 for the \(x^3\) term. I mentioned in class on the first day that every cubic equation can be reduced to an equivalent depressed cubic equation. You will verify that assertion in this exercise:
   The roots of a general cubic equation in \(X\) may be viewed in the \((XY\)-plane) as the intersections of the \(X\)-axis with the graph of a cubic of the form \(Y = X^3 + AX^2 + BX + C\).
   a. Show that the inflection point of the graph occurs at \(X = -\frac{B}{3A}\).
   b. Show that the substitution \(X = \left(x - \frac{B}{3A}\right)\) will reduce the above equation to the form \(Y = x^3 + bx + c\) for some constants \(b\) and \(c\), and determine the values of the constants \(b\) and \(c\) (written in terms of the original constants \(A\), \(B\), and \(C\)).

Extra credit: In the notes, I told you that every depressed cubic equation of the form \(x^3 = 3px + 2q\) has a solution of the form \(x = \sqrt[3]{q + \sqrt{q^2 - p^3}} + \sqrt[3]{q - \sqrt{q^2 - p^3}}\). Verify this fact by carrying out the following steps:
   a. Make the substitution \(x = s + t\), and argue that \(x\) solves the cubic if both \(st = p\) and \(s^3 + t^3 = 2q\).
   b. Eliminate \(t\) between these two equations, thereby obtaining a quadratic equation in \(s^3\).
   c. Solve this quadratic equation to obtain the two possible values of \(s^3\). By symmetry, what are the possible values of \(t^3\)?
   d. Given that \(s^3 + t^3 = 2q\), deduce that \(x = \sqrt[3]{q + \sqrt{q^2 - p^3}} + \sqrt[3]{q - \sqrt{q^2 - p^3}}\) is a solution for the original cubic equation.