**Multivariable Distributions Exercises**

1. Let \( f(x, y) = cx^2y \), \( x = 1, 2, 3 \), and \( y = 1, 2 \) be the joint p.m.f. for \( X \) and \( Y \).
   a. What is the value of the constant \( c \)?
   b. Find the marginal probability mass functions \( f_X \) and \( f_Y \).
   c. Are \( X \) and \( Y \) independent? Explain your answer.
   d. Find \( P(X = Y) \).
   e. Find \( P(X > Y) \).
   f. Find \( E(X), E(Y), E(XY), Cov(X,Y) \), and \( \rho(X,Y) \).

2. Draw 5 cards at random from an ordinary deck of 52 cards. Among these 5 cards, let \( X \) be the number of hearts, and let \( Y \) be the number of spades.
   a. What kind of distribution does \( X \) have? What is the p.m.f. of \( X \)?
   b. Give a formula for the joint p.m.f. of \( X \) and \( Y \). Then make a table showing the values of the joint p.m.f. for all possible pairs \( (x, y) \).
   c. Use the marginal totals of the table from part (b) to find the values of the marginal p.m.f. of \( X \). Verify that these values match the values of the p.m.f. given in part (a).
   d. Are \( X \) and \( Y \) independent? Explain your answer.

3. Suppose that the joint probabilities \( f(x, y) \) of discrete random variables \( X \) and \( Y \) are given in the following table. (So, for example, \( f(-1,0) = 0 \) and \( f(0,0) = 0.4 \).)

\[
\begin{array}{ccc}
X & -1 & 0 & 1 \\
Y & \hline
0 & 0 & 0.4 & 0 \\
1 & 0.3 & 0 & 0.3 \\
\end{array}
\]

   a. Find the marginal distributions of \( X \) and \( Y \).
   b. Show that \( X \) and \( Y \) are NOT independent.
   c. Compute \( E(X), E(Y), E(XY), Cov(X,Y) \), and \( \rho(X,Y) \). Use the result to show that \( X \) and \( Y \) are uncorrelated.
   NOTE: This provides an example to two random variables that are uncorrelated but not independent. It is possible to have two variables that are very strongly related, but still have 0 covariance and 0 correlation. In this case, knowing the value of \( X \) completely determines the value of \( Y \). Based on the table above, if we know the value of \( X \), there is only one possible choice for \( Y \). (In fact, \( Y = X^2 \).)

4. Let \( X \) and \( Y \) have joint probability density function \( f(x, y) = x + y \), \( 0 \leq x \leq 1, 0 \leq y \leq 1 \).
   a. Find the marginal probability density functions \( f_X \) and \( f_Y \).
   b. Are \( X \) and \( Y \) independent? Explain your answer.
   c. Find \( P(X \leq \frac{1}{2} \) and \( Y \geq \frac{1}{2} \)).
   d. Find \( P(X + Y < \frac{1}{2}) \).
   e. Find \( E(X), E(Y), E(XY), Cov(X,Y) \), and \( \rho(X,Y) \).
5. Let $X$ and $Y$ have joint probability density function $f(x, y) = \frac{12}{5} xy(1+y)$, $0 \leq x \leq 1$, $0 \leq y \leq 1$

a. Find $P(\frac{1}{4} \leq X \leq \frac{1}{2}$ and $\frac{1}{3} \leq Y \leq \frac{1}{2}$).

b. Find the marginal probability density functions $f_X(x)$ and $f_Y(y)$.

c. Are $X$ and $Y$ independent? Explain your answer.

d. Without doing any further computations, you should be able to write down the values of $\text{Cov}(X, Y)$ and $\rho(X, Y)$. What are they? Why?

6. Let $X$ and $Y$ have joint p.d.f. $f(x, y) = K \left(3x^2 + 8xy\right)$, $0 \leq x \leq 1$, $0 \leq y \leq 2$.

a. Find $K$.

b. Find $P(2X \leq Y)$.

7. Let $X$ and $Y$ be independent discrete random variables and assume that the moment-generating functions $M_X$ and $M_Y$ both exist. Prove that $M_{X+Y}(t) = M_X(t) \cdot M_Y(t)$. Clearly indicate where the assumption of independence is used in your proof. Hint: To get started, note that by definition, $M_{X+Y}(t) = E \left(e^{(X+Y)t}\right) = \sum_x \sum_y e^{(x+y)t} f(x, y)$.

8. Let $X_1, X_2, X_3, X_4, X_5$ be a random sample from a population with geometric distribution with $p = \frac{1}{4}$.

a. Find the moment generating function of $Y = X_1 + X_2 + X_3 + X_4 + X_5$.

b. What is the distribution of $Y$? What are the mean and variance of $Y$?

9. Let $W = X_1 + X_2 + \cdots + X_n$, where the $X_i$ are independent and identically distributed exponential random variables with parameter $\lambda$. Show that $W$ has a gamma distribution. What are the p.d.f., mean, and variance of $W$?

10. A soft drink company uses a filling machine to fill cans. Each 12 oz. can is to contain 355 milliliters of beverage. In fact, the amount varies according to a normal distribution with mean $\mu = 355.2$ ml and standard deviation $\sigma = 0.5$ ml.

a. What is the probability that an individual can contains less than 355 ml?

b. What is the probability that the mean content of a six-pack of cans is less than 355 ml?

11. The lifetime of disk brake pads varies according to a normal distribution with mean $\mu = 50,000$ miles and standard deviation $\sigma = 3000$ miles. Suppose that a sample of nine brake pads is tested.

a. What is the distribution of the sample mean $\bar{X}$? Give the mean and standard deviation of this distribution.

b. Suppose that a sample mean less than 47,000 miles is considered good evidence that that the true mean lifetime $\mu$ of the break pads is less than 50,000 miles. What is the probability that this will happen (a sample mean of less than 47,000 miles in a sample of size nine) even when the true mean is in fact 50,000 miles, leading to an incorrect conclusion?