2.2 CONTINUITY

In this section we study continuous functions of a real variable. We will prove some important theorems about continuous functions that, although intuitively plausible, are beyond the scope of the elementary calculus course. They are accessible now because of our better understanding of the real number system, especially of those properties that stem from the completeness axiom.

The definitions of

$$f(x_0-) = \lim_{x \to x_0-} f(x), \quad f(x_0+) = \lim_{x \to x_0+} f(x), \quad \text{and} \quad \lim_{x \to x_0} f(x)$$

do not involve $f(x_0)$ or even require that it be defined. However, the case where $f(x_0)$ is defined and equal to one or more of these quantities is important.

Definition 2.2.1

(a) We say that f is *continuous at* x_0 if f is defined on an open interval (a, b) containing x_0 and $\lim_{x \to x_0} f(x) = f(x_0)$.

(b) We say that f is *continuous from the left at* x_0 if f is defined on an open interval (a, x_0) and $f(x_0-) = f(x_0)$.

(c) We say that f is *continuous from the right at* x_0 if f is defined on an open interval (x_0, b) and $f(x_0+) = f(x_0)$. ■

The following theorem provides a method for determining whether these definitions are satisfied. The proof, which we leave to you (Exercise 1), rests on Definitions 2.1.2, 2.1.5, and 2.2.1.

Theorem 2.2.2

(a) *A function f is continuous at x_0 if and only if f is defined on an open interval (a, b) containing x_0 and for each $\epsilon > 0$ there is a $\delta > 0$ such that*

$$|f(x) - f(x_0)| < \epsilon \tag{1}$$

whenever $|x - x_0| < \delta$.

(b) *A function f is continuous from the right at x_0 if and only if f is defined on an interval $[x_0, b)$ and for each $\epsilon > 0$ there is a $\delta > 0$ such that* (1) *holds whenever $x_0 \le x < x_0 + \delta$.*

(c) *A function f is continuous from the left at x_0 if and only if f is defined on an interval $(a, x_0]$ and for each $\epsilon > 0$*

there is a $\delta > 0$ such that (1) *holds whenever $x_0 - \delta < x \le x_0$.*

From Definition 2.2.1 and Theorem 2.2.2, f is continuous at x_0 if and only if

$$f(x_0-) = f(x_0+) = f(x_0)$$

or, equivalently, if and only if it is continuous from the right and left at x_0 (Exercise 2).

Example 2.2.1 Let f be defined on $[0, 2]$ by

$$f(x) = \begin{cases} x^2, & 0 \le x < 1, \\ x + 1, & 1 \le x \le 2 \end{cases}$$

(Figure 2.2.1); then

$$\begin{aligned} f(0+) &= 0 = f(0), \\ f(1-) &= 1 \neq f(1) = 2, \\ f(1+) &= 2 = f(1), \\ f(2-) &= 3 = f(2). \end{aligned}$$

Therefore, f is continuous from the right at 0 and 1 and continuous from the left at 2, but not at 1. If $0 < x, x_0 < 1$, then

$$\begin{aligned} |f(x) - f(x_0)| &= |x^2 - x_0^2| = |x - x_0|\,|x + x_0| \\ &\le 2|x - x_0| < \epsilon \quad \text{if} \quad |x - x_0| < \epsilon/2. \end{aligned}$$

Hence, f is continuous at each x_0 in $(0, 1)$. If $1 < x, x_0 < 2$, then

$$\begin{aligned} |f(x) - f(x_0)| &= |(x + 1) - (x_0 + 1) = |x - x_0| \\ &< \epsilon \quad \text{if} \quad |x - x_0| < \epsilon. \end{aligned}$$

Hence, f is continous at each x_0 in $(1, 2)$. ■

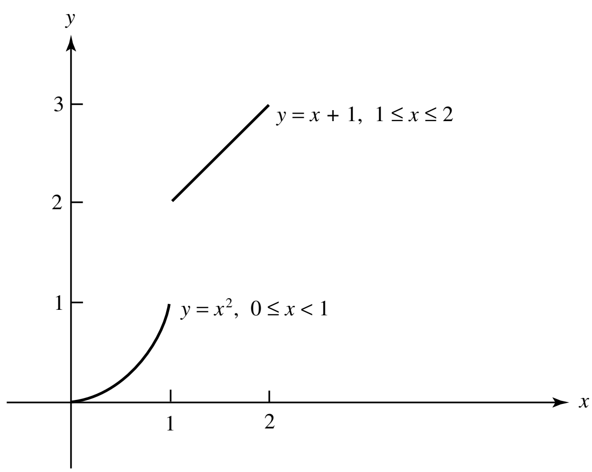

Figure 2.2.1

Definition 2.2.3 A function f is *continuous on an open interval* (a, b) if it is continuous at every point in (a, b). If, in addition,

$$f(b-) = f(b) \tag{2}$$

or

$$f(a+) = f(a) \tag{3}$$

then f is *continuous on* $(a,b]$ *or* $[a,b)$, respectively. If f is continuous on (a,b) and (2) and (3) both hold, then f *is continuous on* $[a,b]$. More generally, if S is a subset of D_f consisting of finitely or infinitely many disjoint intervals, then f is *continuous on* S if f is continuous on every interval in S. (Henceforth, in connection with functions of one variable, whenever we say "f is continuous on S" we mean that S is a set of this kind.)
■

Example 2.2.2 Let $f(x) = \sqrt{x}$, $0 \le x < \infty$. Then

$$|f(x) - f(0)| = \sqrt{x} < \epsilon \quad \text{if} \quad 0 \le x < \epsilon^2,$$

so $f(0+) = f(0)$. If $x_0 > 0$ and $x \ge 0$, then

$$\begin{aligned} |f(x) - f(x_0)| = |\sqrt{x} - \sqrt{x_0}| &= \frac{|x - x_0|}{\sqrt{x} + \sqrt{x_0}} \\ &\le \frac{|x - x_0|}{\sqrt{x_0}} < \epsilon \quad \text{if} \quad |x - x_0| < \epsilon\sqrt{x_0}, \end{aligned}$$

so $\lim_{x \to x_0} f(x) = f(x_0)$. Hence, f is continuous on $[0, \infty)$. ■

Example 2.2.3 The function

$$g(x) = \frac{1}{\sin \pi x}$$

is continuous on $S = \bigcup_{n=-\infty}^{\infty}(n, n+1)$. However, g is not continuous at any $x_0 = n$ (integer), since it is not defined at such points. ■

The function f defined in Example 2.2.1 (see also Figure 2.2.1) is continuous on $[0, 1)$ and $[1, 2]$, but not on any open interval containing 1. The discontinuity of f there is of the simplest kind, described in the following definition.

Definition 2.2.4 A function f is *piecewise continuous* on $[a, b]$ if

(a) $f(x_0+)$ exists for all x_0 in $[a, b)$;

(b) $f(x_0-)$ exists for all x_0 in $(a, b]$;

(c) $f(x_0+) = f(x_0-) = f(x_0)$ for all but finitely many points x_0 in (a, b).

If **(c)** fails to hold at some x_0 in (a, b), f has a *jump discontinuity at* x_0. Also, f has a *jump discontinuity at* a if $f(a+) \ne f(a)$ or *at* b if $f(b-) \ne f(b)$. ■

Example 2.2.4 The function

$$f(x) = \begin{cases} 1, & x = 0, \\ x, & 0 < x < 1, \\ 2, & x = 1, \\ x, & 1 < x \le 2, \\ -1, & 2 < x < 3, \\ 0, & x = 3, \end{cases}$$

(Figure 2.2.2) is the graph of a piecewise continuous function on $[0, 3]$, with jump discontinuities at $x_0 = 0$, 1, 2, and 3. ■

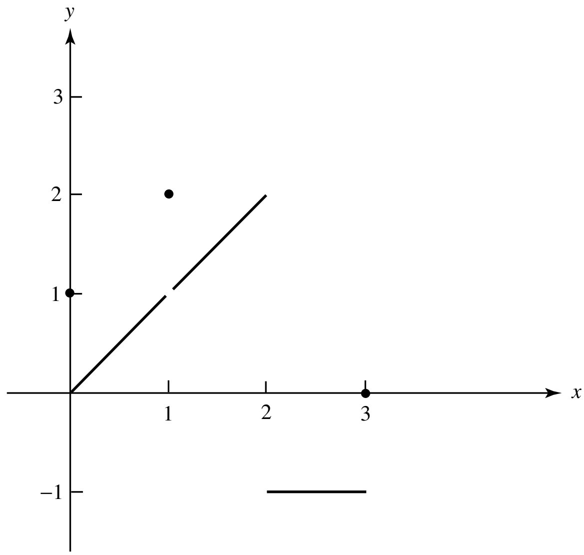

Figure 2.2.2

The reason for the adjective "jump" can be seen in Figures 2.2.1 and 2.2.2, where the graphs exhibit a definite jump at each point of discontinuity. The next example shows that not all discontinuities are of this kind.

Example 2.2.5 The function

$$f(x) = \begin{cases} \sin \dfrac{1}{x}, & x \neq 0, \\ \\ 0, & x = 0, \end{cases}$$

is continuous at all x_0 except $x_0 = 0$. As x approaches 0 from either side, $f(x)$ oscillates between -1 and 1 with ever-increasing frequency, so neither $f(0+)$ nor $f(0-)$ exists. Therefore, the discontinuity of f at 0 is not a jump discontinuity, and if $\rho > 0$, then f is not piecewise continuous on any interval of the form $[-\rho, 0]$, $[-\rho, \rho]$, or $[0, \rho]$. ■

Theorems 2.1.4 and 2.2.2 imply the next theorem (Exercise 18).

Theorem 2.2.5 *If f and g are continuous on a set S, then so are $f + g$, $f - g$, and fg. In addition, f/g is continuous at each x_0 in S such that $g(x_0) \neq 0$.*

Example 2.2.6 Since the constant functions and the function $f(x) = x$ are continuous for all x, successive applications of the various parts of Theorem 2.2.5 imply that the function

$$r(x) = \frac{9 - x^2}{x + 1}$$

is continuous for all x except $x = -1$ (see Example 2.1.7). More generally, by starting from Theorem 2.2.5 and using induction, it can be shown that if $f_1, f_2, \dots, f_n$ are continuous on a set S, then so are $f_1 + f_2 + \cdots + f_n$ and $f_1 f_2 \cdots f_n$. Therefore, any rational function

$$r(x) = \frac{a_0 + a_1 x + \cdots + a_n x^n}{b_0 + b_1 x + \cdots + b_m x^m} \quad (b_m \neq 0)$$

is continuous for all values of x except those for which its denominator vanishes. ■

Removable Discontinuities

Let f be defined on a deleted neighborhood of x_0 and discontinuous (perhaps even undefined) at x_0. We say that f has a *removable discontinuity* at x_0 if $\lim_{x \to x_0} f(x)$ exists. In this case, the function

$$g(x) = \begin{cases} f(x) & \text{if } x \in D_f \text{ and } x \neq x_0, \\ \lim_{x \to x_0} f(x) & \text{if } x = x_0, \end{cases}$$

is continuous at x_0.

Example 2.2.7 The function

$$f(x) = x \sin \frac{1}{x}$$

is not defined at $x_0 = 0$, and therefore certainly not continuous there, but $\lim_{x \to 0} f(x) = 0$ (Example 2.1.6). Therefore, f has a removable discontinuity at 0.

The function

$$f_1(x) = \sin \frac{1}{x}$$

is undefined at 0 and its discontinuity there is not removable, since $\lim_{x \to 0} f_1(x)$ does not exist (Example 2.2.5). ■

Composite Functions

We have seen that the investigation of limits and continuity can be simplified by regarding a given function as the result of addition, subtraction, multiplication, and division of simpler functions. Another operation useful in this connection is *composition* of functions; that is, substitution of one function into another.

Definition 2.2.6 Suppose that f and g are functions with domains D_f and D_g. If D_g has a nonempty subset T such that $g(x) \in D_f$ whenever $x \in T$, then the *composite function* $f \circ g$ is defined on T by

$$(f \circ g)(x) = f(g(x)). \quad ■$$

Example 2.2.8 If

$$f(x) = \log x \quad \text{and} \quad g(x) = \frac{1}{1-x^2},$$

then

$$D_f = (0, \infty) \quad \text{and} \quad D_g = \left\{x \,\middle|\, x \neq \pm 1\right\}.$$

Since $g(x) > 0$ if $x \in T = (-1, 1)$, the composite function $f \circ g$ is defined on $(-1, 1)$ by

$$(f \circ g)(x) = \log \frac{1}{1-x^2}.$$

We leave it to you to verify that $g \circ f$ is defined on $(0, 1/e) \cup (1/e, e) \cup (e, \infty)$ by

$$(g \circ f)(x) = \frac{1}{1-(\log x)^2}. \qquad \blacksquare$$

The next theorem says that the composition of continuous functions is continuous.

Theorem 2.2.7 *Suppose that g is continuous at x_0, $g(x_0)$ is an interior point of D_f, and f is continuous at $g(x_0)$. Then $f \circ g$ is continuous at x_0.*

Proof Suppose that $\epsilon > 0$. Since $g(x_0)$ is an interior point of D_f and f is continuous at $g(x_0)$, there is a $\delta_1 > 0$ such that $f(t)$ is defined and

$$|f(t) - f(g(x_0))| < \epsilon \quad \text{if} \quad |t - g(x_0)| < \delta_1. \tag{4}$$

Since g is continuous at x_0, there is a $\delta > 0$ such that $g(x)$ is defined and

$$|g(x) - g(x_0)| < \delta_1 \quad \text{if} \quad |x - x_0| < \delta. \tag{5}$$

Now (4) and (5) imply that

$$|f(g(x)) - f(g(x_0))| < \epsilon \quad \text{if} \quad |x - x_0| < \delta.$$

Therefore, $f \circ g$ is continuous at x_0. $\square$

See Exercise 22 for a related result concerning limits.

Example 2.2.9 In Examples 2.2.2 and 2.2.6 we saw that the function

$$f(x) = \sqrt{x}$$

is continuous for $x > 0$, and the function

$$g(x) = \frac{9-x^2}{x+1}$$

is continuous for $x \neq -1$. Since $g(x) > 0$ if $x < -3$ or $-1 < x < 3$, Theorem 2.2.7 implies that the function

$$(f \circ g)(x) = \sqrt{\frac{9-x^2}{x+1}}$$

is continuous on $(-\infty, -3) \cup (-1, 3)$. It is also continuous from the left at -3 and 3. $\blacksquare$

Bounded Functions

A function f is *bounded below* on a set S if there is a real number m such that

$$f(x) \geq m \text{ for all } x \in S.$$

In this case, the set

$$V = \{f(x) \,|\, x \in S\}$$

has an infimum α, and we write

$$\alpha = \inf_{x \in S} f(x).$$

If there is a point x_1 in S such that $f(x_1) = \alpha$, we say that α is the *minimum of* f *on* S, and write

$$\alpha = \min_{x \in S} f(x).$$

Similarly, f is *bounded above on* S if there is a real number M such that $f(x) \leq M$ for all x in S. In this case, V has a supremum β, and we write

$$\beta = \sup_{x \in S} f(x).$$

If there is a point x_2 in S such that $f(x_2) = \beta$, we say that β is the *maximum of* f *on* S, and write

$$\beta = \max_{x \in S} f(x).$$

If f is bounded above and below on a set S, we say that f is *bounded* on S.

Figure 2.2.3 illustrates the geometric meaning of these definitions for a function f bounded on an interval $S = [a, b]$. The graph of f lies in the strip bounded by the lines $y = M$ and $y = m$, where M is any upper bound and m is any lower bound for f on $[a, b]$. The narrowest strip containing the graph is the one bounded above by $y = \beta = \sup_{a \leq x \leq b} f(x)$ and below by $y = \alpha = \inf_{a \leq x \leq b} f(x)$.

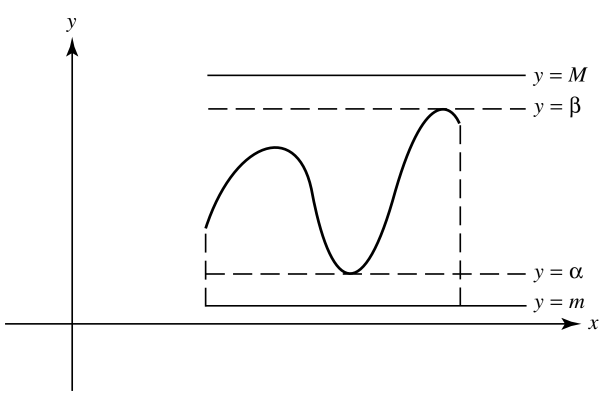

Figure 2.2.3

Example 2.2.10 The function

$$g(x) = \begin{cases} \frac{1}{2}, & x = 0 \quad \text{or} \quad x = 1, \\ 1 - x, & 0 < x < 1, \end{cases}$$

(Figure 2.2.4**(a)**) is bounded on $[0, 1]$, and

$$\sup_{0 \le x \le 1} g(x) = 1, \qquad \inf_{0 \le x \le 1} g(x) = 0.$$

Therefore, g has no maximum or minimum on $[0, 1]$, since it does not assume either of the values 0 and 1.

The function

$$h(x) = 1 - x, \quad 0 \le x \le 1,$$

which differs from g only at 0 and 1 (Figure 2.2.4**(b)**), has the same supremum and infimum as g, but it attains these values at $x = 0$ and $x = 1$, respectively; therefore,

$$\max_{0 \le x \le 1} h(x) = 1 \quad \text{and} \quad \min_{0 \le x \le 1} h(x) = 0.$$ ■

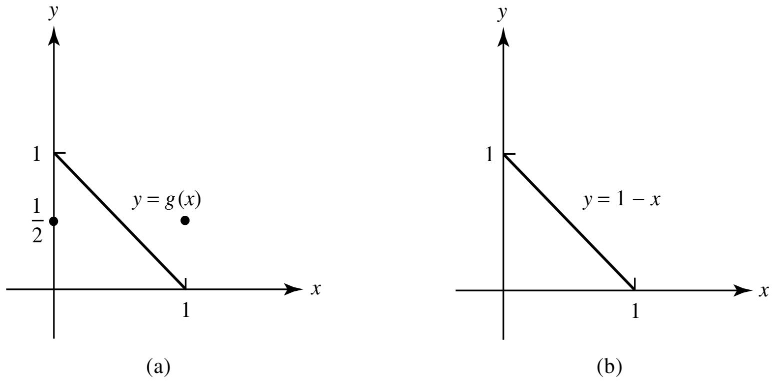

Figure 2.2.4

Example 2.2.11 The function

$$f(x) = e^{x(x-1)} \sin \frac{1}{x(x-1)}, \quad 0 < x < 1,$$

oscillates between $\pm e^{x(x-1)}$ infinitely often in every interval of the form $(0, \rho)$ or $(1-\rho, 1)$, where $0 < \rho < 1$, and

$$\sup_{0 < x < 1} f(x) = 1, \qquad \inf_{0 < x < 1} f(x) = -1.$$

However, f does not assume these values, so f has no maximum or minimum on $(0, 1)$. ■

Theorem 2.2.8 *If f is continuous on a finite closed interval $[a, b]$, then f is bounded on $[a, b]$.*

Proof Suppose that $t \in [a, b]$. Since f is continuous at t, there is an open interval I_t containing t such that

$$|f(x) - f(t)| < 1 \quad \text{if} \quad x \in I_t \cap [a, b]. \tag{6}$$

(To see this, set $\epsilon = 1$ in (1), Theorem 2.2.2.) The collection $H = \left\{ I_t \,\middle|\, a \le t \le b \right\}$ is an open covering of $[a, b]$. Since $[a, b]$ is compact, the Heine–Borel theorem implies that there are finitely many points $t_1, t_2, \dots, t_n$ such that the intervals $I_{t_1}, I_{t_2}, \dots, I_{t_n}$ cover $[a, b]$. According to (6) with $t = t_i$,

$$|f(x) - f(t_i)| < 1 \quad \text{if} \quad x \in I_{t_i} \cap [a, b].$$

Therefore,

$$\begin{aligned} |f(x)| = |(f(x) - f(t_i)) + f(t_i)| &\le |f(x) - f(t_i)| + |f(t_i)| \\ &\le 1 + |f(t_i)| \quad \text{if} \quad x \in I_{t_i} \cap [a, b]. \end{aligned} \tag{7}$$

Let

$$M = 1 + \max_{1 \le i \le n} |f(t_i)|.$$

Since $[a, b] \subset \bigcup_{i=1}^{n} \left(I_{t_i} \cap [a, b] \right)$, (7) implies that $|f(x)| \le M$ if $x \in [a, b]$. □

This proof illustrates the utility of the Heine–Borel theorem, which allows us to choose M as the largest of a *finite* set of numbers.

Theorem 2.2.8 and the completeness of the reals imply that

if f is continuous on a finite closed interval $[a, b]$, then f has an infimum and a supremum on $[a, b]$. The next theorem shows that f actually assumes these values at some points in $[a, b]$.

Theorem 2.2.9 *Suppose that f is continuous on a finite closed interval $[a, b]$. Let*

$$\alpha = \inf_{a \le x \le b} f(x) \quad \textit{and} \quad \beta = \sup_{a \le x \le b} f(x).$$

Then α and β are respectively the minimum and maximum of f on $[a, b]$; that is, there are points x_1 and x_2 in $[a, b]$ such that

$$f(x_1) = \alpha \quad \textit{and} \quad f(x_2) = \beta.$$

Proof We show that x_1 exists and leave it to you to show that x_2 exists (Exercise 24).

Suppose that there is no x_1 in $[a, b]$ such that $f(x_1) = \alpha$. Then $f(x) > \alpha$ for all $x \in [a, b]$. We will show that this leads to a contradiction.

Suppose that $t \in [a,b]$. Then $f(t) > \alpha$, so

$$f(t) > \frac{f(t)+\alpha}{2} > \alpha.$$

Since f is continuous at t, there is an open interval I_t about t such that

$$f(x) > \frac{f(t)+\alpha}{2} \quad \text{if} \quad x \in I_t \cap [a,b] \tag{8}$$

(Exercise 15). The collection $H = \left\{ I_t \,\middle|\, a \le t \le b \right\}$ is an open covering of $[a,b]$. Since $[a,b]$ is compact, the Heine–Borel theorem implies that there are finitely many points t_1, $t_2, \dots, t_n$ such that the intervals $I_{t_1}, I_{t_2}, \dots, I_{t_n}$ cover $[a,b]$. Define

$$\alpha_1 = \min_{1 \le i \le n} \frac{f(t_i)+\alpha}{2}.$$

Then, since $[a,b] \subset \bigcup_{i=1}^{n} (I_{t_i} \cap [a,b])$, (8) implies that

$$f(t) > \alpha_1, \quad a \le t \le b.$$

But $\alpha_1 > \alpha$, so this contradicts the definition of α. Therefore, $f(x_1) = \alpha$ for some x_1 in $[a,b]$. □

Example 2.2.12 We used the compactness of $[a,b]$ in the proof of Theorem 2.2.9 when we invoked the Heine–Borel theorem. To see that compactness is essential to the proof, consider the function

$$g(x) = 1 - (1-x)\sin\frac{1}{x},$$

which is continuous and has supremum 2 on the noncompact interval (0, 1], but does not assume its supremum on (0, 1], since

$$\begin{aligned} g(x) &\le 1 + (1-x)\left|\sin\frac{1}{x}\right| \\ &\le 1 + (1-x) < 2 \quad \text{if} \quad 0 < x \le 1. \end{aligned}$$

As another example, consider the function

$$f(x) = e^{-x},$$

which is continuous and has infimum 0, which it does not attain, on the noncompact interval $(0, \infty)$. ■

The next theorem shows that if f is continuous on a finite closed interval $[a,b]$, then f assumes every value between $f(a)$ and $f(b)$ as x varies from a to b (Figure 2.2.5, page 64).

Theorem 2.2.10 (Intermediate Value Theorem) *Suppose that f is continuous on $[a,b]$, $f(a) \ne f(b)$, and μ is between $f(a)$ and $f(b)$. Then $f(c) = \mu$ for some c in (a,b).*

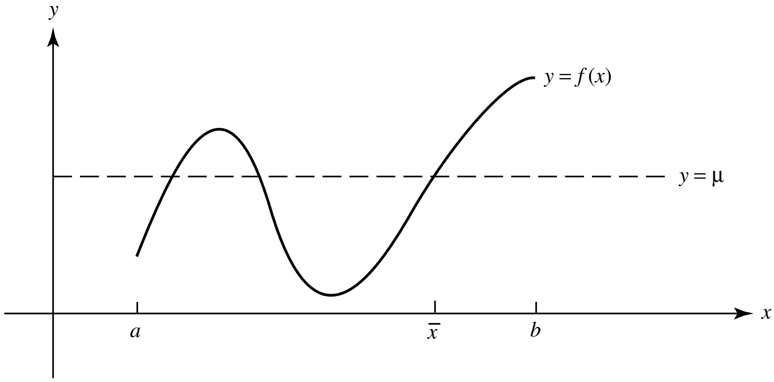

Figure 2.2.5

Proof Suppose that $f(a) < \mu < f(b)$. The set

$$S = \left\{x \,\middle|\, a \le x \le b \quad \text{and} \quad f(x) \le \mu\right\}$$

is bounded and nonempty. Let $c = \sup S$. We will show that $f(c) = \mu$. If $f(c) > \mu$, then $c > a$ and, since f is continuous at c, there is an $\epsilon > 0$ such that $f(x) > \mu$ if $c - \epsilon < x \le c$ (Exercise 15). Therefore, $c - \epsilon$ is an upper bound for S, which contradicts the definition of c as the supremum of S. If $f(c) < \mu$, then $c < b$ and there is an $\epsilon > 0$ such that $f(x) < \mu$ for $c \le x < c + \epsilon$, so c is not an upper bound for S. This is also a contradiction. Therefore, $f(c) = \mu$.

The proof for the case where $f(b) < \mu < f(a)$ can be obtained by applying this result to $-f$. ☐

Uniform Continuity

Theorem 2.2.2 and Definition 2.2.3 imply that a function f is continuous on a subset S of its domain if for each $\epsilon > 0$ and each x_0 in S, there is a $\delta > 0$, *which may depend upon* x_0 *as well as* ϵ, such that

$$|f(x) - f(x_0)| < \epsilon \quad \text{if} \quad |x - x_0| < \delta \quad \text{and} \quad x \in D_f.$$

The next definition introduces another kind of continuity on a set S.

Definition 2.2.11 A function f is *uniformly continuous* on a subset S of its domain if, for every $\epsilon > 0$, there is a $\delta > 0$ such that

$$|f(x) - f(x')| < \epsilon \text{ whenever } |x - x'| < \delta \text{ and } x, x' \in S. \qquad \blacksquare$$

We emphasize that in this definition δ depends only on ϵ and S and not on the particular choice of x and x', provided that they are both in S.

Example 2.2.13 The function

$$f(x) = 2x$$

is uniformly continuous on $(-\infty, \infty)$, since

$$|f(x) - f(x')| = 2|x - x'| < \epsilon \quad \text{if} \quad |x - x'| < \epsilon/2. \qquad \blacksquare$$

Example 2.2.14 If $0 < r < \infty$, then the function

$$g(x) = x^2$$

is uniformly continuous on $[-r, r]$. To see this, note that

$$|g(x) - g(x') = |x^2 - (x')^2| = |x - x'|\,|x + x'| \le 2r|x - x'|,$$

so

$$|g(x) - g(x')| < \epsilon \quad \text{if} \quad |x - x'| < \delta = \frac{\epsilon}{2r} \quad \text{and} \; -r \le x, x' \le r. \qquad \blacksquare$$

Often a concept is clarified by considering its negation: a function f is *not* uniformly continuous on S if there is an $\epsilon_0 > 0$ such that if δ is any positive number, there are points x and x' in S such that

$$|x - x'| < \delta \quad \text{but} \quad |f(x) - f(x')| \ge \epsilon_0.$$

Example 2.2.15 The function $g(x) = x^2$ is uniformly continuous on $[-r, r]$ for any finite r (Example 2.2.14), but not on $(-\infty, \infty)$. To see this, we will show that if $\delta > 0$ there are real numbers x and x' such that

$$|x - x'| = \delta/2 \quad \text{and} \quad |g(x) - g(x')| \ge 1.$$

To this end, we write

$$|g(x) - g(x')| = |x^2 - (x')^2| = |x - x'|\,|x + x'|.$$

If $|x - x'| = \delta/2$ and $x, x' > 1/\delta$, then

$$|x - x'|\,|x + x'| > \frac{\delta}{2}\left(\frac{1}{\delta} + \frac{1}{\delta}\right) = 1. \qquad \blacksquare$$

Example 2.2.16 The function

$$f(x) = \cos\frac{1}{x}$$

is continuous on $(0, 1]$ (Exercise 23**(i)**). However, f is not uniformly continuous on $(0, 1]$, since

$$\left| f\left(\frac{1}{n\pi}\right) - f\left(\frac{1}{(n+1)\pi}\right)\right| = 2, \quad n = 1, 2, \dots. \qquad \blacksquare$$

Examples 2.2.15 and 2.2.16 show that a function may be continuous but not uniformly continuous on an interval. The next theorem shows that this cannot happen if the interval is closed and bounded, and therefore compact.

Theorem 2.2.12 *If f is continuous on a closed and bounded interval $[a,b]$, then f is uniformly continuous on $[a,b]$.*

Proof Suppose that $\epsilon > 0$. Since f is continuous on $[a,b]$, for each t in $[a,b]$ there is a positive number δ_t such that

$$|f(x) - f(t)| < \frac{\epsilon}{2} \quad \text{if} \quad |x - t| < 2\delta_t \quad \text{and} \quad x \in [a,b]. \tag{9}$$

If $I_t = (t - \delta_t, t + \delta_t)$, the collection

$$H = \left\{ I_t \,\middle|\, t \in [a,b] \right\}$$

is an open covering of $[a,b]$. Since $[a,b]$ is compact, the Heine–Borel theorem implies that there are finitely many points $t_1, t_2, \dots, t_n$ in $[a,b]$ such that $I_{t_1}, I_{t_2}, \dots, I_{t_n}$ cover $[a,b]$. Now define

$$\delta = \min\{\delta_{t_1}, \delta_{t_2}, \dots, \delta_{t_n}\}. \tag{10}$$

We will show that if

$$|x - x'| < \delta \quad \text{and} \quad x, x' \in [a,b], \tag{11}$$

then $|f(x) - f(x')| < \epsilon$.

From the triangle inequality,

$$\begin{aligned} |f(x) - f(x')| &= |\,(f(x) - f(t_r)) + (f(t_r) - f(x'))\,| \\ &\le |f(x) - f(t_r)| + |f(t_r) - f(x')|. \end{aligned} \tag{12}$$

Since $I_{t_1}, I_{t_2}, \dots, I_{t_n}$ cover $[a,b]$, x must be in one of these intervals. Suppose that $x \in I_{t_r}$; that is,

$$|x - t_r| < \delta_{t_r}. \tag{13}$$

From (9) with $t = t_r$,

$$|f(x) - f(t_r)| < \frac{\epsilon}{2}. \tag{14}$$

From (11), (13), and the triangle inquality,

$$|x' - t_r| = |(x' - x) + (x - t_r)| \le |x' - x| + |x - t_r| < \delta + \delta_{t_r} \le 2\delta_{t_r}.$$

Therefore, (9) with $t = t_r$ and x replaced by x' implies that

$$|f(x') - f(t_r)| < \frac{\epsilon}{2}.$$

This, (12), and (14) imply that $|f(x) - f(x')| < \epsilon$. □

This proof again shows the utility of the Heine–Borel theorem, which allowed us to define δ in (10) as the smallest of a *finite* set of positive numbers, so that δ is sure to be positive. (An infinite set of positive numbers may fail to have a smallest positive member; for example, consider the open interval (0, 1).)

Corollary 2.2.13 *If f is continuous on a set T, then f is uniformly continuous on any finite closed interval contained in T.*

Applied to Example 2.2.16, Corollary 2.2.13 implies that the function $g(x) = \cos 1/x$ is uniformly continuous on $[\rho, 1]$ if $0 < \rho < 1$.

More About Monotonic Functions

Theorem 2.1.9 implies that if f is monotonic on an interval I, then f is either continuous or has a jump discontinuity at each x_0 in I. This and Theorem 2.2.10 provide the key to the proof of the following theorem.

Theorem 2.2.14 *If f is monotonic and nonconstant on $[a, b]$, then f is continuous on $[a, b]$ if and only if its range $R_f = \left\{f(x) \,\middle|\, x \in [a, b]\right\}$ is the closed interval with endpoints $f(a)$ and $f(b)$.*

Proof We assume that f is nondecreasing, and leave the case where f is nonincreasing to you (Exercise 34). Theorem 2.1.9**(a)** implies that the set $\widetilde{R}_f = \left\{f(x) \,\middle|\, x \in (a, b)\right\}$ is a subset of the open interval $(f(a+), f(b-))$. Therefore,

$$R_f = \{f(a)\} \cup \widetilde{R}_f \cup \{f(b)\} \subset \{f(a)\} \cup (f(a+), f(b-)) \cup \{f(b)\}. \qquad (15)$$

Now suppose that f is continuous on $[a, b]$. Then $f(a) = f(a+)$, $f(b-) = f(b)$, so (15) implies that $R_f \subset [f(a), f(b)]$. If $f(a) < \mu < f(b)$, then Theorem 2.2.10 implies that $\mu = f(x)$ for some x in (a, b). Hence, $R_f = [f(a), f(b)]$.

For the converse, suppose that $R_f = [f(a), f(b)]$. Since $f(a) \le f(a+)$ and $f(b-) \le f(b)$, (15) implies that $f(a) = f(a+)$ and $f(b-) = f(b)$. We know from Theorem 2.1.9**(c)** that if f is nondecreasing and $a < x_0 < b$, then

$$f(x_0-) \le f(x_0) \le f(x_0+).$$

If either of these inequalities is strict, R_f cannot be an interval. Since this contradicts our assumption, $f(x_0-) = f(x_0) = f(x_0+)$. Therefore, f is continuous at x_0 (Exercise 2). We can now conclude that f is continuous on $[a, b]$. □

Theorem 2.2.14 implies the following theorem.

Theorem 2.2.15 *Suppose that f is increasing and continuous on $[a, b]$, and let $f(a) = c$ and $f(b) = d$. Then there is a unique function g defined on $[c, d]$ such that*

$$g(f(x)) = x, \quad a \le x \le b, \qquad (16)$$

and

$$f(g(y)) = y, \quad c \le y \le d. \qquad (17)$$

Moreover, g is continuous and increasing on $[c, d]$.

Proof We first show that there is a function g satisfying (16) and (17). Since f is continuous, Theorem 2.2.14 implies that for each y_0 in $[c, d]$ there is an x_0 in $[a, b]$ such that

$$f(x_0) = y_0, \qquad (18)$$

and, since f is increasing, there is only one such x_0. Define

$$g(y_0) = x_0. \tag{19}$$

The definition of x_0 is illustrated in Figure 2.2.6: with $[c, d]$ drawn on the y-axis, find the intersection of the line $y = y_0$ with the curve $y = f(x)$ and drop a vertical from the intersection to the x-axis to find x_0.

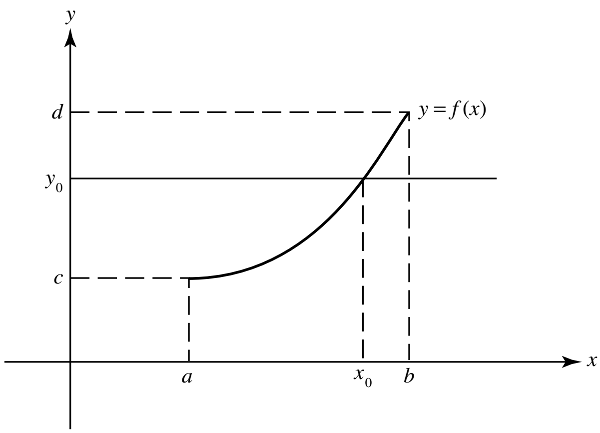

Figure 2.2.6

Substituting (19) into (18) yields

$$f(g(y_0)) = y_0,$$

and substituting (18) into (19) yields

$$g(f(x_0)) = x_0.$$

Dropping the subscripts in these two equations yields (16) and (17).

The uniqueness of g follows from our assumption that f is increasing, and therefore only one value of x_0 can satisfy (18) for each y_0.

To see that g is increasing, suppose that $y_1 < y_2$ and let x_1 and x_2 be the points in $[a, b]$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Since f is increasing, $x_1 < x_2$. Therefore,

$$g(y_1) = x_1 < x_2 = g(y_2),$$

so g is increasing. Since $R_g = \left\{ g(y) \,\middle|\, y \in [c, d] \right\}$ is the interval $[g(c), g(d)] = [a, b]$, Theorem 2.2.14 with f and $[a, b]$ replaced by g and $[c, d]$ implies that g is continuous on $[c, d]$. $\square$

The function g of Theorem 2.2.15 is the *inverse* of f, denoted by f^{-1}. Since (16) and (17) are symmetric in f and g, we can also regard f as the inverse of g, and denote it by g^{-1}.

Example 2.2.17 If

$$f(x) = x^2, \quad 0 \le x \le R,$$

then

$$f^{-1}(y) = g(y) = \sqrt{y}, \quad 0 \le y \le R^2. \qquad \blacksquare$$

Example 2.2.18 If

$$f(x) = 2x + 4, \quad 0 \le x \le 2,$$

then

$$f^{-1}(y) = g(y) = \frac{y-4}{2}, \quad 4 \le y \le 8. \qquad \blacksquare$$