Finite-horizon interest rate

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If the annual interest rate is $r$, then a value of $A$ dollars today can be converted to a value of $A(1+r)$ dollars one year from now, a value of $A(1+r)^2$ dollars two years from now, and $A(1+r)^n$ dollars $n$ years from now.

Reversing this, a value of $A$ dollars $n$ years from now has the equivalent value to

$$\frac{A}{(1+r)^n}$$

dollars today. Since investment makes a dollar today worth more than a dollar in the future, a given dollar amount farther in the future is worth less than that amount today. The value today is called the *present value*, and is given by the formula above.

If an investment pays $X$ dollars every year for $n$ years, its present value is:

$$PV = \frac{X}{1+r} + \frac{X}{(1+r)^2} + \frac{X}{(1+r)^3} + \ldots + \frac{X}{(1+r)^n}$$

Let $u = \frac{1}{1+r}$:

$$PV = X(u + u^2 + u^3 + \ldots + u^n)$$

Multiply both sides by $1 - u$:

$$(1-u)PV = X(u + u^2 + u^3 + \ldots + u^n)(1-u)$$

$$(1-u)PV = X[(u + u^2 + u^3 + \ldots + u^n) - (u^2 + u^3 + \ldots + u^{n+1})]$$

$$(1-u)PV = X(u - u^{n+1})$$

$$PV = \frac{X(u - u^{n+1})}{1-u}$$

$$PV = \frac{X \left( \frac{1}{1+r} - \left( \frac{1}{1+r} \right)^{n+1} \right)}{1 - \frac{1}{1+r}}$$
Multiply the top and bottom of the fraction by \((1 + r)\):

\[
PV = \frac{X \left(1 - \left(\frac{1}{1+r}\right)^n\right)}{1 + r - 1}
\]

\[
PV = \frac{X \left(1 - \left(\frac{1}{1+r}\right)^n\right)}{r}
\]

Multiply the top and bottom of the fraction by \((1 + r)^n\):

\[
PV = \frac{X[(1 + r)^n - 1]}{r(1 + r)^n}
\]

Define the finite-horizon interest rate as:

\[
r'_{n} = r \left( \frac{(1 + r)^n}{(1 + r)^n - 1} \right)
\]

Then the present value formula becomes:

\[
PV = \frac{X}{r'_{n}}
\]

When \(n\) is very large, then \((1 + r)^n - 1\) is almost the same as \((1 + r)^{n}\), so \(r'_{n} = r\). For a very large time horizon, the finite horizon interest rate becomes the same as the regular interest rate.

For example, if the interest rate is \(r = 0.06\), the finite-horizon-20-year rate is

\[
r'_{20} = 0.06 \left( \frac{1.06^{20}}{1.06^{20} - 1} \right)
\]

\[
= 0.06 \left( \frac{3.2071}{3.2071 - 1} \right)
\]

\[
= 0.0872
\]

If you take out a loan for \(A\) dollars and make interest payments forever, without ever paying back the principal, your interest payment each period will be \(rA\). If the loan is arranged so you make equal payments each period, but pay back the entire loan amount after \(n\) periods (as with a traditional fixed-rate home mortgage), then the amount you pay each period is \(r'_{n}A\).
The finite horizon rate \( r'_n \) will be slightly larger than the regular interest rate \( r \). The difference will be smaller when the repayment period is longer.

In a traditional home mortgage loan, payments are made every month. The annual interest rate \( r \) needs to be divided by 12 to get the monthly rate, and then there are \( n = 360 \) periods. If the annual interest rate is 0.060 (6.0 percent), the monthly rate is 0.005, the 360-month finite-horizon rate is

\[
r'_{360} = 0.0050 \left( \frac{1.0050^{360}}{1.0050^{360} - 1} \right)
\]

\[
= 0.00599551
\]

and the monthly payment for a 300,000 dollar loan is

\[
0.00599551 \times 300,000 = 1,798.65
\]