Complex Variables: Homework #1

Read http://math.fullerton.edu/mathews/c2003/ComplexNumberOrigin.html.

1. Verify that $2-\sqrt{-1}$ is a cube root of $2-11\sqrt{-1}$.

2. Explain why cubic equations, rather than quadratic equations, played the pivotal role in helping to obtain the acceptance of complex numbers.

3. Find all solutions to $27x^3-9x-2=0$ by performing the following steps:
   i. Determine values of p and q such that this equation is equivalent to an equation of the form $x^3=3px+2q$.
   ii. Use the formula from slide #5 from the first day of class to find one solution to the equation.
   iii. Use the root found in (ii) to factor the original cubic into a linear term and a quadratic term. Find the roots (if any) of the quadratic term.

4. Use Bombelli's techniques to find all solutions to $x^3-87x-130=0$. (For an example of similar computations, look at the example of Bombelli's "wild thought" about midway through the webpage listed at the top of this assignment.)

5. A **depressed cubic equation** is a cubic equation which (i) has no x^2 term and (ii) has a coefficient of 1 for the x^3 term. I mentioned in class on the first day that *every* cubic equation can be reduced to an equivalent depressed cubic equation. You will verify that assertion in this exercise:
   The roots of a general cubic equation in X may be viewed in the (XY-plane) as the intersections of the X-axis with the graph of a cubic of the form $Y=X^3+AX^2+BX+C$.
   a. Show that the inflection point of the graph occurs at $X=-\frac{A}{3}$.
   b. Show that the substitution $X=\left(x-\frac{A}{3}\right)$ will reduce the above equation to the form $Y=x^3+bx+c$ for some constants b and c, and determine the values of the constants b and c.

Extra credit: In the notes, I told you that every depressed cubic equation of the form $x^3=3px+2q$ has a solution of the form $x=\sqrt[3]{q+\sqrt{q^2-p^3}}+\sqrt[3]{q-\sqrt{q^2-p^3}}$. Verify this fact by carrying out the following steps:
   a. Make the substitution $x=s+t$, and argue that x solves the cubic if both $st=p$ and $s^3+t^3=2q$.
   b. Eliminate t between these two equations, thereby obtaining a quadratic equation in s^3.
   c. Solve this quadratic equation to obtain the two possible values of s^3. By symmetry, what are the possible values of t^3?
   d. Given that $s^3+t^3=2q$, deduce that $x=\sqrt[3]{q+\sqrt{q^2-p^3}}+\sqrt[3]{q-\sqrt{q^2-p^3}}$ is a solution for the original cubic equation.