

Why i ?

The Historical Roots of “Imaginary” Numbers

MAT 3730: Complex Variables
Brian Gill

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The first appearance in print of a square root of a negative number:

Gerolamo Cardano’s *Ars Magna*, 1545

In *Ars Magna*, Cardano poses the following problem: “divide 10 into two parts, one of which multiplied into the other shall produce 40.”

Cardano noted that the solution was impossible.

But then he went on to say, “Nevertheless, we shall solve it in this fashion” and gave as solutions

$$5 + \sqrt{-15} \quad \text{and} \quad 5 - \sqrt{-15}$$

Cardano’s “Useless” Arithmetic

Problem: “Divide 10 into two parts, one of which multiplied into the other shall produce 40.”

Are $5 + \sqrt{-15}$ and $5 - \sqrt{-15}$ reasonable solutions?

Adding them together, we obtain

$$(5 + \sqrt{-15}) + (5 - \sqrt{-15}) = 5 + 5 = 10$$

so he had “divided 10 into two parts”.

Furthermore, “putting aside the tortures involved” and using the familiar rules for multiplication, we obtain

$$\begin{aligned} (5 + \sqrt{-15}) \times (5 - \sqrt{-15}) \\ &= 25 - 5\sqrt{-15} + 5\sqrt{-15} - (\sqrt{-15})^2 \\ &= 25 - (-15) \\ &= 40 \end{aligned}$$

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Cardano's "Useless" Arithmetic

But what is meant by $\sqrt{-15}$?? Is it even a "number"? Cardano himself was more confused than convinced by his solution, calling it "useless" and saying "so progresses arithmetic subtlety, the end of which is as refined as it is useless".

Cardano's "Useless" Arithmetic

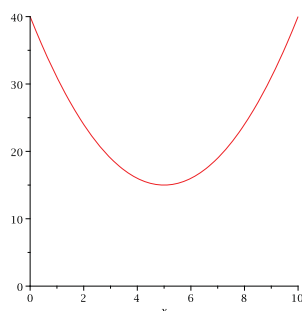
Note: Cardano's problem is equivalent to looking for solutions to the quadratic equation $x(10 - x) = 40$, which can be rewritten $x^2 - 10x + 40 = 0$. Although they did not use notation that even remotely resembles our modern notation, it was known by 2000 BC that solutions to a quadratic equation $ax^2 + bx + c = 0$ were given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

It was also well understood that when $b^2 - 4ac < 0$, the equation had **no** solutions.

Cardano's "Useless" Arithmetic

The quadratic formula for Cardano's problem *does* yield the "solutions" $5 + \sqrt{-15}$ and $5 - \sqrt{-15}$, BUT *no mathematician at the time cared at all about these solutions*. It was quite clear from the geometry of the problem that there were NO solutions, so these "solutions" *must* be absolute nonsense!



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Negative Numbers: “*numeri absurdi*”

Although it may seem strange to us, mathematicians of the time even had serious difficulty with the notion of negative numbers. Many considered negative numbers (let alone their square roots!) to be imaginary. Michael Stifel called *negative* numbers “*numeri absurdi*” and Cardano referred to them as “*numeri ficti*.” Even Descartes referred to negatives as “false roots”.

Negative Numbers

As late as 1759, Baron Francis Masères had the following to say about negative numbers:

“They serve only, as far as I am able to judge, to puzzle the whole doctrine of equations, and to render obscure and mysterious things that are in their own nature exceedingly plain and simple. . . It were to be wished therefore that negative roots had never been admitted into algebra or were again discarded from it: for if this were done, there is good reason to imagine, the objections which many learned and ingenious men now make to algebraic computations, as being obscure and perplexed with almost unintelligible notions, would be thereby removed.”

Cubic Equations

In the 16th century, it was a very big deal to try to find a formula similar to the quadratic formula that would work for *cubic* equations – this was one of the great mathematical problems of the time.

It was known that every cubic equation could be reduced to an equation of the form

$$x^3 = 3px + 2q$$

where p and q are real numbers. Furthermore, it was known that *every* such equation had at least one solution.

Again in *Ars Magna*, Cardano published the first solution for a cubic equation of the form above. In modern notation, a solution for this equation is

$$x = \sqrt[3]{q + \sqrt{q^2 - p^3}} + \sqrt[3]{q - \sqrt{q^2 - p^3}}.$$

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Cubic Equations: An Example

A solution to $x^3 = 3px + 2q$ is

$$x = \sqrt[3]{q + \sqrt{q^2 - p^3}} + \sqrt[3]{q - \sqrt{q^2 - p^3}}.$$

For example, Cardano's formula can be used to find a solution to the cubic equation $x^3 = -24x + 56$. With $p = -8$ and $q = 28$, we have

$$\sqrt{q^2 - p^3} = \sqrt{28^2 - (-8)^3} = \sqrt{784 + 512} = \sqrt{1296} = 36.$$

So a solution for the cubic is

$$x = \sqrt[3]{28 + 36} + \sqrt[3]{28 - 36} = \sqrt[3]{64} + \sqrt[3]{-8} = 4 - 2 = 2$$

Cubic Equations: Cardano's Formula Fails?

But there was a major difficulty with his solution:

Consider the equation $x^3 = 15x + 4$.

With this cubic, we have

$$\sqrt{q^2 - p^3} = \sqrt{2^2 - 5^3} = \sqrt{4 - 125} = \sqrt{-121}.$$

Once again we have encountered the dreaded square root of a negative number. But this time there is a difference:

With the quadratic equation, when we get a negative in the radical, it means that there are NO real solutions to the equation. On the other hand, the cubic above DOES have 3 real solutions, but it appears that Cardano's formula fails to find any of them!

So matters stood for nearly 30 years.

Rafael Bombelli's "wild thought" (1572)

Throwing aside momentarily any prejudice about square roots of negative numbers, write

$$\sqrt{-121} = \sqrt{121 \times (-1)} = \sqrt{121} \times \sqrt{-1} = 11\sqrt{-1}$$

Now Cardano's formula for the equation $x^3 = 15x + 4$ becomes

$$x = \sqrt[3]{2 + 11\sqrt{-1}} + \sqrt[3]{2 - 11\sqrt{-1}}$$

Applying what seemed like reasonable rules of arithmetic, Bombelli then noted that

$$\begin{aligned} (2 + \sqrt{-1})^3 &= (4 + 4\sqrt{-1} + (\sqrt{-1})^2) (2 + \sqrt{-1}) \\ &= (3 + 4\sqrt{-1}) (2 + \sqrt{-1}) \\ &= 6 + 11\sqrt{-1} + 4(\sqrt{-1})^2 \\ &= 2 + 11\sqrt{-1} \end{aligned}$$

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Rafael Bombelli's "wild thought" (1572)

In other words,

$$\sqrt[3]{2 + 11\sqrt{-1}} = 2 + \sqrt{-1}$$

A similar computation shows that

$$\sqrt[3]{2 - 11\sqrt{-1}} = 2 - \sqrt{-1}$$

So finally we have that

$$\begin{aligned} x &= \sqrt[3]{2 + 11\sqrt{-1}} + \sqrt[3]{2 - 11\sqrt{-1}} \\ &= (2 + \sqrt{-1}) + (2 - \sqrt{-1}) = 4 \end{aligned}$$

is a solution to the cubic equation

$$x^3 = 15x + 4.$$

Rafael Bombelli's "wild thought" (1572)

Note that once we have **one** solution, we can find any additional solutions to the cubic equation by factoring it to reduce it to a quadratic. For example, rewriting $x^3 = 15x + 4$ as $x^3 - 15x - 4 = 0$ and factoring out $(x - 4)$ yields $(x - 4)(x^2 + 4x + 1)$. The remaining two roots can be found using the quadratic formula:

$$x = (-4 \pm \sqrt{16 - 4})/2 = (-4 \pm 2\sqrt{3})/2 = -2 \pm \sqrt{3}.$$

We started with a problem involving real numbers and ended with a real solution. But solving this perfectly real problem required the use of complex arithmetic!

Rafael Bombelli's "wild thought" (1572)

Problems like the one in the example above appear very frequently when trying to use Cardano's formula for solutions of a cubic equation. In order to deal with this problem, Bombelli then attempted to find a general method for reducing expressions of the form $\sqrt[3]{a + b\sqrt{-1}}$ (where a and b are real numbers) to the form $c + d\sqrt{-1}$ (where c and d are real numbers). But alas, he failed.

Unable to find a general method and forced to rely purely on specific examples like the one above, he called his approach "a wild thought" and said that "the whole matter seems to rest on sophistry rather than on truth."

It would take two more centuries, the development of calculus, and the genius of Leonhard Euler to deal with this question.

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“Imaginary” Numbers

60 years later Descartes in his *Géométrie* said “Neither the true nor the false [negative] roots are always real; sometimes they are imaginary.” This seems to be the source of the term “imaginary” numbers. (The term “real” number was later introduced precisely to distinguish them from “imaginary” numbers.)

Still later Newton called these numbers “impossible”. “But it is just that the Roots of Equations should be often impossible, lest they should exhibit the cases of Problems that are impossible as if they were possible.”

In 1702, Leibniz referred to $\sqrt{-1}$ as “that amphibian between being and non-being which we call the imaginary root of negative unity.”

Euler’s Insight

Finally, in the 1740’s, the insight of Leonhard Euler moved imaginary numbers several huge steps closer to full partnership in mathematics.

First there was a small notational contribution: it was Euler that helped standardize the use of the symbol i (for imaginary) to represent $\sqrt{-1}$.

Perhaps Euler’s greatest contribution to complex (imaginary) numbers was the following formula for all real numbers θ :

$$e^{i\theta} = \cos \theta + i \sin \theta.$$

What could possibly make him write down such a strange formula?

Euler’s Insight

Recall from Calculus: for all *real* numbers x ,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

Also note that

$$i^2 = (\sqrt{-1})^2 = -1$$

$$i^3 = i^2 i = (-1)i = -i$$

$$i^4 = i^3 i = (-i)i = -i^2 = -(-1) = 1$$

$$i^5 = i^4 i = i$$

etc.

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Euler's Insight

Recall from Calculus: for all *real* numbers x ,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

Also, $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $i^5 = i, \dots$

Temporarily putting aside questions of convergence, if we replace the real number x with the imaginary number $i\theta$ in the power series, we obtain

$$\begin{aligned} e^{i\theta} &= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \cdots \\ &= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} - \frac{\theta^6}{6!} - i\frac{\theta^7}{7!} + \cdots \\ &= \left[1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots \right] + i \left[\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots \right] \\ &= \cos \theta + i \sin \theta \end{aligned}$$

Euler's Insight

But Euler did not stop there...

Based on his formula, huge strides were made in developing the arithmetic of complex numbers. Among other things, he went on to apply his formula along with the standard rules of arithmetic for real numbers to find an imaginary power of an imaginary number.

From his formula, we have

$e^{i\pi/2} = \cos(\pi/2) + i \sin(\pi/2) = i$. Along with the familiar rule $(a^r)^s = a^{rs}$ for exponents, this yields

$$i^i = (e^{i\pi/2})^i = e^{-\pi/2} = \frac{1}{\sqrt{e^\pi}}.$$

An imaginary power of an imaginary number gives a *real* number!

Unfortunately, even Euler did not fully understand the arithmetic of complex numbers. In one of his writings, he made the subtle error of stating that $\sqrt{-3}\sqrt{-2} = \sqrt{6}$. This *seems* reasonable (just extend the identity $\sqrt{ab} = \sqrt{a}\sqrt{b}$ from real to imaginary numbers), but leads to many inconsistencies and contradictions. For example:

$$1 = \sqrt{1} = \sqrt{(-1)(-1)} = \sqrt{-1}\sqrt{-1} = (i)(i) = i^2 = -1.$$

In fact, the product of two square roots of negative numbers should be computed as follows:

$$\sqrt{-3}\sqrt{-2} = (i\sqrt{3})(i\sqrt{2}) = i^2\sqrt{6} = -\sqrt{6}.$$

What went wrong? Euler did not fully grasp the subtleties of extending the definition of the square root function to negative numbers. In addition, it seems clear that he did not have a clear picture of the *geometry* involved in complex arithmetic.

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The Complex Plane

By around 1800, several mathematicians independently began to view **complex numbers** as points in the plane: that is, a complex number is a number of the form $z = x + iy$ and z can be represented as the point (x, y) in the plane. Euler's equation helped considerably both in leading to this geometric viewpoint and in the understanding of the basic arithmetic of complex numbers. By representing a point (x, y) in polar form as $(r \cos \theta, r \sin \theta)$, and then applying Euler's formula, we obtain

$$z = x + iy = r \cos \theta + ir \sin \theta = r(\cos \theta + i \sin \theta) = re^{i\theta}.$$

So by converting a point into polar coordinates, we obtain an exponential representation for each complex number. From this viewpoint, multiplication, division, and roots of complex numbers can be naturally defined using standard rules for exponentials.

An Example: Complex Roots

As an example, let's try to compute $\sqrt[3]{-1 + i}$. It is very hard to do directly...

But, geometrically, this is just the point $(-1, 1)$ in the plane. In polar coordinates, this can be represented by $r = \sqrt{2}$ and $\theta = 3\pi/4$. But then

$$-1 + i = \sqrt{2} \cos(3\pi/4) + i\sqrt{2} \sin(3\pi/4) = \sqrt{2}e^{i3\pi/4}.$$

Now comes the magic of Euler's formula:

$$\begin{aligned}\sqrt[3]{-1 + i} &= (-1 + i)^{1/3} = \left(\sqrt{2}e^{i3\pi/4}\right)^{1/3} \\ &= \left(2^{1/2}\right)^{1/3} \left(e^{i3\pi/4}\right)^{1/3} \\ &= \sqrt[6]{2}e^{i\pi/4} = \sqrt[6]{2}(\cos(\pi/4) + i \sin(\pi/4)) \\ &= \sqrt[6]{2} \left(\frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}}\right) = \frac{2^{1/6}}{2^{1/2}}(1 + i) = 2^{-1/3}(1 + i)\end{aligned}$$

At about the same time as Euler's work, d'Alembert made a significant discovery in hydrodynamics. Clairaut had previously noted that in a two dimensional fluid flow, the two components P and Q of the velocity of the flow must satisfy

$$\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} = 0$$

in order to be a steady, irrotational flow. D'Alembert discovered that a second relation,

$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = 0,$$

represented the incompressibility of the fluid. These two conditions together in fact imply that the complex function $P + iQ$ has a derivative with respect to the variable z . They would become the foundation for the theory of complex functions developed in the 19th century by Cauchy and Riemann.

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The fundamental theorem of algebra

Every nonconstant polynomial with complex coefficients has a root in the complex number system.

Suppose $f(z)$ is a polynomial of degree $n > 0$ with complex coefficients and leading coefficient d . Then there are n (not necessarily distinct) complex numbers c_1, c_2, \dots, c_n such that

$$f(z) = d(z - c_1)(z - c_2) \cdots (z - c_n).$$

This theorem establishes the complex numbers as the algebraist's paradise. The work of Euler, d'Alembert, and later of Cauchy established the central role of complex numbers in function theory, physics, and in calculus. Complex numbers had truly come of age.

Finally, we return to a special case of Euler's formula. It is

one of the most famous equations in all of mathematics.

$$e^{i\pi} + 1 = 0$$

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