The Gamma Function

Work in small groups (preferably 3 people per group) on the following problems. At the end of class your group should turn in ONE copy of your work with everyone's names on it. If you have not completed all of the problems, just turn in whatever you have finished. Then complete and turn in the other problems by Thursday, either individually or in groups (preferably as a group, but if you can't get together with the others in your group, just do it on your own).

Def: The ***gamma function***, denoted $\Gamma(t)$, is defined by

$$\Gamma(t) = \int_0^\infty x^{t-1} e^{-x} dx \qquad \text{for } t > 0.$$

1. Compute $\Gamma(1)$.

2. Use integration by parts to show that $\Gamma(t) = (t-1)\cdot\Gamma(t-1)$.

3. Using the results of problems 1 and 2, show that for any positive integer n, $\Gamma(n) = (n-1)!$ For this reason, the Gamma function is often called the ***generalized factorial function***. Note that $(n-1)!$ is only defined for positive integers, but the gamma function is defined for all positive real numbers t.

4. Using a suitable change of variable in the integral defining the gamma function, show that
$$\Gamma(t) = 2^{1-t} \cdot \int_0^\infty y^{2t-1} e^{-\frac{1}{2}y^2} dy \qquad \text{for } t > 1.$$

5. Using the formula from problem 4, show that
$$\Gamma\left(\tfrac{1}{2}\right) = \sqrt{2}\int_0^\infty e^{-\frac{1}{2}y^2} dy .$$

6. As a consequence of problem 5,
$$\left[\Gamma\left(\tfrac{1}{2}\right)\right]^2 = \left[\sqrt{2}\int_0^\infty e^{-\frac{1}{2}y^2} dy\right]\cdot\left[\sqrt{2}\int_0^\infty e^{-\frac{1}{2}x^2} dx\right] = 2\int_0^\infty\int_0^\infty e^{-\frac{1}{2}\left(x^2+y^2\right)} dxdy.$$
Convert the double integral above into polar coordinates and then evaluate the integral to show that
$$\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}.$$

7. The p.d.f. for the *normal distribution* is of the form
$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \quad -\infty < x < \infty \text{ (where } \mu \text{ and } \sigma \text{ are constants and } \sigma > 0\text{)}$$
Show that $\int_{-\infty}^{\infty} f(x)\,dx = 1$. Hint: Substitute $y = \dfrac{x-\mu}{\sigma}$ and then use the symmetry of f to convert to an integral from 0 to infinity.

8. A random variable X has the *gamma distribution* with parameters λ and r if its p.d.f. is given by
$$g(x) = \frac{\lambda^r}{\Gamma(r)} x^{r-1} e^{-\lambda x}, \quad 0 \le x < \infty,$$
where λ and r are positive real numbers. Show that $\int_{-\infty}^{\infty} g(x)\,dx = 1$.